

Comparing localizations across adjunctions

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Abstract

We show that several apparently unrelated formulas involving localizations in homotopy theory are special cases of comparison maps associated with a pair of adjoint functors. In doing so, we find that some of these formulas can be generalized and obtain new ones. Among other results, we prove that if T is a monad acting on a model category then the localization of a homotopy T -algebra X with respect to a map f is equivalent to the Tf -localization of X under very general assumptions.

1 Introduction

Localization with respect to a map has been extensively studied in homotopy theory and also in group theory; informative references are [12, 22, 24, 28, 34]. This article arose from the observation that a number of distinct formulas in this context follow a common pattern not previously revealed. As a motivation to start with, compare the following facts:

- (a) In the category of groups, $L_f A \cong L_{f_{\text{ab}}} A$ for every homomorphism f and every abelian group A , where the subindex “ab” denotes abelianization.
- (b) In the homotopy category of simplicial sets, $L_f X \simeq L_{SP^\infty f} X$ for every map f and every simplicial abelian group X , where SP^∞ denotes the infinite symmetric product [23].
- (c) In the stable homotopy category, $L_f X \simeq L_{H\mathbb{Z} \wedge f} X$ for every map f and every spectrum X that splits as a product of Eilenberg–Mac Lane spectra. Here $H\mathbb{Z}$ denotes the spectrum that represents ordinary homology with integral coefficients.

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Each of these formulas is linked with preservation of some sort of structure by localizations. References are given in the article. For example, the fact that every localization in the category of groups preserves commutativity yields a natural group homomorphism

$$\alpha: (LG)_{\text{ab}} \longrightarrow L(G_{\text{ab}})$$

for every group G . Although this homomorphism is almost never an isomorphism, it induces an isomorphism

$$L((LG)_{\text{ab}}) \cong L(G_{\text{ab}}) \quad (1.1)$$

for all groups G . In connection with this, statement (a) above is proved by using the fact that both L_f and $L_{f_{\text{ab}}}$ restrict to abelian groups and checking that their essential image classes coincide.

In Section 5 we prove that, in analogy with (a), for every ring R with 1 we have

$$L_f M \cong L_{R \otimes f} M \quad (1.2)$$

for each group homomorphism f between abelian groups and every left R -module M . Here L_f is meant as a localization on the category of abelian groups and $L_f M$ really means $L_f U M$, where U is the forgetful functor from R -modules to abelian groups. Similarly, $L_{R \otimes f} M$ should be read as $L_{U(R \otimes f)} U M$. The proof of (1.2) depends crucially on the fact that localization functors on groups send R -modules to R -modules.

Localizations of groups can be extended over groupoids and then one has, for every map f of spaces, a natural homomorphism of groupoids

$$\alpha: \pi L_f X \longrightarrow L_{\pi f}(\pi X)$$

which is an isomorphism only in special cases, yet it induces an isomorphism

$$L_{\pi f}(\pi L_f X) \cong L_{\pi f}(\pi X)$$

for every space X , similarly as in (1.1). This was proved in [18] using the fact that, although localizations of groupoids can be described in discrete categorical terms, they fit very naturally in the context of homotopical localizations on model categories.

One of our main results in this article is an extension of (1.2) to the homotopical context, namely

$$L_f M \simeq L_{E \wedge f} M \quad (1.3)$$

if E is any strict ring spectrum (i.e., a monoid in some structured model category of spectra) that is cofibrant as a spectrum, while M is an E -module spectrum and f is any map of cofibrant spectra such that L_f commutes with suspension. The same result holds without the latter restriction on f if the spectrum E is connective.

Unexpectedly, however, (1.3) still holds if ring spectra and module spectra are meant in the homotopical sense. In fact, stable homotopical localizations preserve homotopy module spectra by [19] and also strict module spectra

by [20]. It then turns out that (1.3) generalizes [31, Proposition 3.2]. Note that statement (c) above is a special case.

This led us to study preservation of algebras over monads by localizations in model categories, not only in the strict sense (that is, when the given monad acts on the model category and the resulting category of algebras admits a transferred model structure) but also in the homotopical sense (by letting the monad act on the homotopy category, assuming that it preserves weak equivalences). We found that the general formula

$$L_f X \simeq L_{Tf} X \quad (1.4)$$

holds for homotopy T -algebras whenever the monad T preserves f -equivalences and Tf -equivalences, and this condition is automatically satisfied if T is the (reduced or unreduced) monad associated with a unital operad acting on pointed simplicial sets.

Our approach is based on a study of *comparison maps* of type

$$\alpha: FL_f \longrightarrow L_{Ff}F \quad \text{or} \quad \beta: L_f G \longrightarrow GL_{Ff} \quad (1.5)$$

made in the first sections of this paper for each Quillen pair of adjoint functors F and G (this is an instance of a pair of *mates* in the sense of [47]). Such comparison maps arise very frequently, and are equivalences in some cases. The following formula was proved by Farjoun in [24]:

$$L_f \Omega X \simeq \Omega L_{\Sigma f} X, \quad (1.6)$$

and the next one was proved by Bousfield in [11] for nullification functors P_W with respect to a space W :

$$P_W \Omega^\infty X \simeq \Omega^\infty P_{\Sigma^\infty W} X. \quad (1.7)$$

These formulas are “of β type” as depicted in (1.5). In Section 7 we show that, as one would surely expect, (1.7) holds for arbitrary f -localizations, not only nullifications. The proofs of (1.6) and (1.7) proceed in two steps, namely one proves first that L_f sends loop spaces to loop spaces and infinite loop spaces to infinite loop spaces using one’s choice of higher homotopy techniques, and then simple categorical arguments are brought into play to infer the given formulas. Our purpose in this article is to distill general formal arguments from ad hoc facts behind each of these formulas.

As special cases of (1.4), we obtain that

$$L_f X \simeq L_{\Omega \Sigma f} X \quad \text{and} \quad L_f X \simeq L_{\Omega^\infty \Sigma^\infty f} X,$$

assuming in each case that X is a homotopy algebra over the corresponding monad, namely $\Omega \Sigma$ in the first case and $Q = \Omega^\infty \Sigma^\infty$ in the second case. The second one can be generalized as

$$L_f X \simeq L_{\Omega^\infty(E \wedge \Sigma^\infty f)} X$$

for homotopy algebras over the monad $X \mapsto \Omega^\infty(E \wedge \Sigma^\infty X)$, where E is any connective ring spectrum. By choosing $E = H\mathbb{Z}$ one recovers statement (b) from the beginning of this Introduction.

In the last section of the article we state, without proofs, analogous facts and formulas for colocalizations and cellular approximations.

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2 Adjunctions and monads

This section contains standard terminology and a few basic facts that will be used in the article. More information about adjunctions and monads can be found e.g. in [39, Chapters IV and VI].

If \mathcal{C} and \mathcal{D} are categories, we denote by $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a pair of adjoint functors, with F left adjoint and G right adjoint, meaning that there are natural bijections of morphism sets

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY) \quad (2.1)$$

for all X in \mathcal{C} and Y in \mathcal{D} . We denote by $\varphi^t : X \rightarrow GY$ the adjoint of a morphism $\varphi : FX \rightarrow Y$ under (2.1), and, similarly, $\psi^t : FX \rightarrow Y$ denotes the adjoint of a morphism $\psi : X \rightarrow GY$. Adjoints of identities yield natural transformations $\eta : \text{Id} \rightarrow GF$ (the *unit*) and $\varepsilon : FG \rightarrow \text{Id}$ (the *counit*), which in turn determine the adjunction by $\psi^t = \varepsilon_Y \circ F\psi$ and $\varphi^t = G\varphi \circ \eta_X$.

If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a pair of adjoint functors, then F is a retract of FGF and G is a retract of GFG . This follows from the triangle identities

$$\varepsilon_{FX} \circ F\eta_X = (\eta_X)^t = \text{id}_{FX} \quad \text{and} \quad G\varepsilon_Y \circ \eta_{GY} = (\varepsilon_Y)^t = \text{id}_{GY} \quad (2.2)$$

for all X and Y . Moreover, if we consider the full subcategories

$$\mathcal{C}_\eta = \{X \in \mathcal{C} \mid \eta_X : X \cong GFX\}, \quad \mathcal{D}_\varepsilon = \{Y \in \mathcal{D} \mid \varepsilon_Y : FGY \cong Y\},$$

then F and G restrict to an equivalence of categories

$$F : \mathcal{C}_\eta \rightleftarrows \mathcal{D}_\varepsilon : G. \quad (2.3)$$

A *monad* on a category \mathcal{C} is a triple (T, η, μ) where $T : \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $\eta : \text{Id} \rightarrow T$ and $\mu : TT \rightarrow T$ are natural transformations such that

$$\mu \circ T\mu = \mu \circ \mu T \quad \text{and} \quad \mu \circ T\eta = \mu \circ \eta T = \text{Id}_T.$$

A monad (T, η, μ) is called *idempotent* if μ is an isomorphism, which we then omit from the notation. If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is an adjunction, then $(GF, \eta, G\varepsilon F)$ is a monad. In fact, all monads are of this form, in more than one way.

If (T, η, μ) is a monad on a category \mathcal{C} , then a *T-algebra* is a pair (X, a) with $a : TX \rightarrow X$ such that

$$a \circ Ta = a \circ \mu_X \quad \text{and} \quad a \circ \eta_X = \text{id}_X.$$

A morphism of T -algebras $(X, a) \rightarrow (Y, b)$ is a morphism $\varphi: X \rightarrow Y$ in \mathcal{C} such that $\varphi \circ a = b \circ T\varphi$. Thus, the T -algebras form a category \mathcal{C}^T , called the *Eilenberg–Moore category* of T , which is equipped with an adjunction

$$F: \mathcal{C} \rightleftarrows \mathcal{C}^T: U, \quad (2.4)$$

where $FX = (TX, \mu_X)$ and U is the forgetful functor. This adjunction is terminal among those whose associated monad is T .

A full subcategory \mathcal{S} of a category \mathcal{C} is *reflective* if the inclusion I is part of an adjunction $K: \mathcal{C} \rightleftarrows \mathcal{S}: I$. Then the functor $L = IK$ is called a *reflection* or a *localization* on \mathcal{C} . In this case, the counit $KI \rightarrow \text{Id}$ is an isomorphism. We denote the unit by $l: \text{Id} \rightarrow L$. Thus, (L, l) is an idempotent monad. An object of \mathcal{C} is called *L -local* if it is isomorphic to an object in the subcategory \mathcal{S} ; hence, X is L -local if and only if $l_X: X \rightarrow LX$ is an isomorphism. A morphism $g: U \rightarrow V$ is an *L -equivalence* if Lg is an isomorphism, or, equivalently, if for all L -local objects X composition with g induces a bijection

$$\mathcal{C}(V, X) \cong \mathcal{C}(U, X). \quad (2.5)$$

Conversely, the L -local objects are precisely those X for which (2.5) holds for all L -equivalences $g: U \rightarrow V$; see [2] for further details.

3 Comparison morphisms

Suppose given a localization L_1 on a category \mathcal{C}_1 and a localization L_2 on a category \mathcal{C}_2 . We say that a functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ *preserves local objects* if FX is L_2 -local for every L_1 -local object X , and we say that F *preserves equivalences* if Ff is an L_2 -equivalence whenever f is an L_1 -equivalence.

Proposition 3.1. *Let $F: \mathcal{C}_1 \rightleftarrows \mathcal{C}_2: G$ be a pair of adjoint functors. Let L_1 be a localization on \mathcal{C}_1 and L_2 a localization on \mathcal{C}_2 . Then G preserves local objects if and only if F preserves equivalences.*

Proof. This follows from the definitions, using the natural bijection

$$\mathcal{C}_2(Ff, X) \cong \mathcal{C}_1(f, GX)$$

for a map f in \mathcal{C}_1 and an object X in \mathcal{C}_2 . □

Theorem 3.2. *Let $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a functor. Let L_1 be a localization on \mathcal{C}_1 with unit l_1 , and L_2 a localization on \mathcal{C}_2 with unit l_2 . Then the following hold:*

- (i) *F preserves equivalences if and only if there is a natural transformation*

$$\alpha: FL_1 \longrightarrow L_2F$$

such that $\alpha \circ Fl_1 = l_2F$ and α_X is an L_2 -equivalence for all X in \mathcal{C}_2 . If this holds, then α is unique, and it is an isomorphism if and only if F preserves local objects.

(ii) F preserves local objects if and only if there is a natural transformation

$$\beta: L_2F \longrightarrow FL_1$$

such that $\beta \circ l_2F = FL_1$. If this holds, then β is unique, and it is an isomorphism if and only if F preserves equivalences.

Proof. For every X in \mathcal{C}_2 , the morphism $(l_1)_X: X \rightarrow L_1X$ is an L_1 -equivalence. Therefore, if F preserves equivalences, then $F(l_1)_X$ is an L_2 -equivalence. Hence it induces a natural bijection

$$\mathcal{C}_2(FL_1X, L_2FX) \xrightarrow{\cong} \mathcal{C}_2(FX, L_2FX)$$

and α_X is uniquely defined by the equality $\alpha_X \circ F(l_1)_X = (l_2)_{FX}$. Since $(l_2)_{FX}$ and $F(l_1)_X$ are both L_2 -equivalences, α_X is also an L_2 -equivalence. In order to prove that α is a natural transformation, we need to check that $\alpha_Y \circ FL_1f$ is equal to $L_2Ff \circ \alpha_X$ for every $f: X \rightarrow Y$. But this follows from the equality

$$L_2Ff \circ \alpha_X \circ F(l_1)_X = \alpha_Y \circ FL_1f \circ F(l_1)_X,$$

using the fact that $F(l_1)_X$ is an L_2 -equivalence and L_2FY is L_2 -local.

Now assume that α exists with the given properties. Then the equality $\alpha_X \circ F(l_1)_X = (l_2)_{FX}$ implies that $F(l_1)_X$ is an L_2 -equivalence for all X . From this fact it follows that F preserves equivalences, for if $f: U \rightarrow V$ is an L_1 -equivalence, then FL_1f is an isomorphism, and, since $FL_1f \circ F(l_1)_U = F(l_1)_V \circ Ff$, we infer that Ff is an L_2 -equivalence.

If F preserves equivalences, then α exists, and α_X is an isomorphism if and only if FL_1X is L_2 -local, since every L_2 -equivalence between L_2 -local objects is an isomorphism. This completes the proof of part (i).

We omit the proof of part (ii), which is similar. \square

Example 3.3. For a cocomplete category \mathcal{C} and a small category I , choose F to be the colimit functor

$$\operatorname{colim}_I \mathcal{C}^I \longrightarrow \mathcal{C}.$$

Let L be a localization on \mathcal{C} . Extend it objectwise over \mathcal{C}^I ; that is, for a diagram $X: I \rightarrow \mathcal{C}$, define $(LX)_i = L(X_i)$ for all $i \in I$. Thus, the diagonal functor $\mathcal{C} \rightarrow \mathcal{C}^I$ preserves local objects and therefore the colimit functor preserves equivalences by Proposition 3.1. Therefore Theorem 3.2 yields a natural L -equivalence

$$\alpha: \operatorname{colim}_{i \in I} LX_i \longrightarrow L\left(\operatorname{colim}_{i \in I} X_i\right) \quad (3.1)$$

which is an L -equivalence, that is,

$$L\left(\operatorname{colim}_{i \in I} LX_i\right) \simeq L\left(\operatorname{colim}_{i \in I} X_i\right).$$

This is an instance of the well-known fact that left adjoints preserve colimits (by viewing L as a left adjoint to the inclusion of the full subcategory of L -local objects into \mathcal{C}).

The diagonal functor $\mathcal{C} \rightarrow \mathcal{C}^I$ also preserves equivalences, and hence, if \mathcal{C} is complete, then the limit functor preserves local objects. This yields a natural morphism

$$L\left(\lim_{i \in I} X_i\right) \longrightarrow \lim_{i \in I} LX_i,$$

which is rarely an equivalence of any type.

Corollary 3.4. *Let $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a functor. Let L_1 be a localization on \mathcal{C}_1 and L_2 a localization on \mathcal{C}_2 . Then L_2F and FL_1 are naturally isomorphic if and only if F preserves local objects and equivalences. In this case, $\alpha: FL_1 \rightarrow L_2F$ and $\beta: L_2F \rightarrow FL_1$ are mutually inverse isomorphisms.*

Proof. The “if” part follows from part (ii) of Theorem 3.2. For the converse, note that $L_2F \cong FL_1$ implies that F preserves local objects, and the naturality of the isomorphism adds the fact that F preserves equivalences, since, for a morphism f , we have that L_2Ff is an isomorphism if and only if FL_1f is an isomorphism. Furthermore, if F preserves local objects and equivalences, then the equality $\alpha \circ \beta \circ l_2F = l_2F$ implies that $\alpha \circ \beta = \text{id}$. \square

4 Induced localizations on algebras

Let L_1 be a localization on a category \mathcal{C}_1 and L_2 a localization on a category \mathcal{C}_2 . We say that a functor $G: \mathcal{C}_2 \rightarrow \mathcal{C}_1$ *reflects local objects* if, for an object X of \mathcal{C}_2 , the assertion that GX is L_1 -local implies that X is L_2 -local. Similarly, G *reflects equivalences* if f is an L_2 -equivalence whenever Gf is an L_1 -equivalence.

Proposition 4.1. *Let \mathcal{S} be a full subcategory of a category \mathcal{C} and let L be a localization on \mathcal{C} . If L preserves \mathcal{S} then L restricts to a localization on \mathcal{S} , and the inclusion $I: \mathcal{S} \rightarrow \mathcal{C}$ preserves and reflects local objects and equivalences.*

Proof. Consider the full subcategory \mathcal{L} of \mathcal{S} consisting of all L -local objects of \mathcal{C} that are in \mathcal{S} . Then, for each object X in \mathcal{S} , the morphism $l_X: X \rightarrow LX$ is in \mathcal{S} by assumption, and, for each Y in \mathcal{L} , it induces a bijection

$$\mathcal{S}(LX, Y) = \mathcal{C}(LX, Y) \cong \mathcal{C}(X, Y) = \mathcal{S}(X, Y),$$

so L restricts indeed to a reflection of \mathcal{S} onto \mathcal{L} such that the inclusion preserves and reflects local objects. If $f: X \rightarrow Y$ is a morphism in \mathcal{S} , then Lf is an isomorphism in \mathcal{S} if and only if it is an isomorphism in \mathcal{C} , since I reflects isomorphisms. Hence, I also preserves and reflects equivalences. \square

The following result enhances [22, Theorem 1.2].

Theorem 4.2. *Let (T, η, μ) be a monad on a category \mathcal{C} and let (L, l) be a localization on \mathcal{C} . Then the following statements are equivalent:*

- (a) *T preserves L -equivalences.*
- (b) *For every T -algebra (X, a) there is a unique T -algebra structure on LX such that $l_X: X \rightarrow LX$ is a morphism of T -algebras.*

(c) *There is a localization L' on the category \mathcal{C}^T of T -algebras such that $LU \cong UL'$ naturally, where U denotes the forgetful functor.*

(d) *There is a localization L' on the category \mathcal{C}^T of T -algebras such that the forgetful functor U preserves and reflects local objects and equivalences.*

Proof. We first show that (a) \Rightarrow (b). In order to obtain $\tilde{a}: TLX \rightarrow LX$, use the fact that LX is L -local and Tl_X is an L -equivalence by assumption. Full details can be found in the proof of [22, Theorem 1.2].

Next we prove that (b) \Rightarrow (c). For each T -algebra (X, a) , let us define $L'(X, a) = (LX, \tilde{a})$, where \tilde{a} is given by assumption. Thus,

$$LU(X, a) = LX = UL'(X, a)$$

for all X and all $a: TX \rightarrow X$. To check that L' is indeed a localization, suppose given any morphism $g: (X, a) \rightarrow (Y, b)$ of T -algebras, and suppose further that Y is L -local. Then there is a unique morphism $g': LX \rightarrow Y$ in \mathcal{C} such that $g' \circ l_X = g$. We just need to prove that g' is also a morphism of T -algebras, that is, that $g' \circ \tilde{a}$ is equal to $b \circ Tg'$. This follows from the fact that $Tl_X: TX \rightarrow TLX$ is an L -equivalence and Y is L -local, since

$$b \circ Tg' \circ Tl_X = b \circ Tg = g \circ a = g' \circ l_X \circ a = g' \circ \tilde{a} \circ Tl_X.$$

Now suppose that (c) holds. Then Corollary 3.4 tells us that U preserves local objects and equivalences. To prove that U reflects local objects, suppose that $U(X, a)$ is L -local. Then l_X is an isomorphism. Since α_X is also an isomorphism by Corollary 3.4, we infer that $Ul'_{(X, a)}$ is an isomorphism. Since U reflects isomorphisms, $l'_{(X, a)}$ is an isomorphism, so (X, a) is L' -local, as needed. The fact that U reflects equivalences follows from the equality $UL'f \circ \beta_{(X, a)} = \beta_{(Y, b)} \circ LUf$ for every $f: (X, a) \rightarrow (Y, b)$, together with the fact that U reflects isomorphisms.

Finally, we prove that (d) \Rightarrow (a). For this, just write $T = UF$ and note that U preserves equivalences and F also preserves equivalences by Proposition 3.1. \square

When the equivalent conditions of Theorem 4.2 are satisfied, we say that L' is *induced* by L . Note that, if the monad (T, η, μ) is idempotent, then the conditions of Theorem 4.2 are in their turn equivalent to the condition that L preserves the class of T -local objects, and in this case the induced localization L' is the restriction of L to this class.

If L' is induced by L , then there is a unique natural transformation

$$\alpha: FL \longrightarrow L'F$$

under F such that α_X is an L' -equivalence for all X . This follows from part (i) of Theorem 3.2.

Example 4.3. If L is any localization on the category of groups, then, as shown in [16, Theorem 2.2], L preserves abelian groups. This yields a natural group homomorphism

$$\alpha_G: (LG)_{\text{ab}} \longrightarrow L(G_{\text{ab}}) \quad (4.1)$$

and a natural isomorphism

$$L((LG)_{\text{ab}}) \cong L(G_{\text{ab}})$$

for all groups G and every localization L . We note, however, that (4.1) is far from being an isomorphism in general. For instance, if L is localization at a set of primes P and F is a free group of rank n , then $(F_P)_{\text{ab}} \cong (\mathbb{Z}_P)^n \oplus T$ where T is a big P' -torsion group, as shown by Baumslag in [6]. Thus $(F_P)_{\text{ab}}$ is not P -local, although $((F_P)_{\text{ab}})_P \cong (F_{\text{ab}})_P$.

Example 4.4. Let \mathcal{C} be any closed monoidal category with internal hom equal to $\mathcal{C}(-, -)$, such as the category of abelian groups. If R is any monoid in \mathcal{C} , then $TA = R \otimes A$ defines a monad, whose algebras are the left R -modules. Then T preserves L -equivalences for every localization L on \mathcal{C} , since, if $f: X \rightarrow Y$ is an L -equivalence and Z is L -local, we have

$$\mathcal{C}(Tf, Z) = \mathcal{C}(R \otimes f, Z) \cong \mathcal{C}(R, \mathcal{C}(f, Z)),$$

which is an isomorphism. Consequently, every localization on \mathcal{C} induces a localization on the category of left R -modules, with the property that the forgetful functor preserves and reflects local objects and equivalences. This result improves and generalizes [22, Theorem 4.3].

We remark that the condition that T preserves L -equivalences holds for all localizations L whenever $\mathcal{C}(T-, -)$ depends functorially on $\mathcal{C}(-, -)$, as in the previous example.

5 Inverting one morphism

The main source of motivation of the present article is the study of localizations of the form L_f for a single morphism f , which we next discuss. An object X and a morphism $f: A \rightarrow B$ in a category \mathcal{C} are called *orthogonal* if the function

$$\mathcal{C}(f, X): \mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X) \quad (5.1)$$

is a bijection. The objects orthogonal to a given morphism f are called *f -local* and the morphisms orthogonal to all f -local objects are called *f -equivalences*. An *f -localization* of an object X is an f -equivalence into an f -local object $l_X: X \rightarrow L_f X$. If an f -localization exists for all objects, then L_f is indeed a localization on \mathcal{C} . As shown in [1], the existence of L_f is ensured for all morphisms f if the category \mathcal{C} is locally presentable.

Theorem 5.1. *Let $F: \mathcal{C}_1 \rightleftarrows \mathcal{C}_2: G$ be a pair of adjoint functors. For a morphism $f: A \rightarrow B$ in \mathcal{C}_1 , the following assertions hold:*

- (i) *An object Y in \mathcal{C}_2 is Ff -local if and only if GY is f -local.*
- (ii) *F sends f -equivalences to Ff -equivalences.*

(iii) If L_f and L_{Ff} exist, then there is a unique natural transformation

$$\alpha: FL_f \longrightarrow L_{Ff}F$$

such that $\alpha_X \circ Fl_X = l_{FX}$ and α_X is an Ff -equivalence for all X . There is also a unique natural transformation

$$\beta: L_fG \longrightarrow GL_{Ff}$$

such that $\beta_Y \circ l_{GY} = Gl_Y$ for all Y . Furthermore, α is an isomorphism if and only if F preserves local objects, and β is an isomorphism if and only if G preserves equivalences.

Proof. For any object Y of \mathcal{C}_2 , consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_1(B, GY) & \xrightarrow{C_1(f, GY)} & \mathcal{C}_1(A, GY) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{C}_2(FB, Y) & \xrightarrow{C_2(Ff, Y)} & \mathcal{C}_2(FA, Y), \end{array}$$

where the vertical bijections are given by the adjunction. It follows that GY is f -local if and only if Y is Ff -local, as claimed. Thus, assuming that L_f and L_{Ff} exist, G preserves and reflects local objects, F preserves equivalences by Proposition 3.1, and all the claims in part (iii) follow from Theorem 3.2. \square

The natural transformations α and β given in part (iii) of Theorem 5.1 are *mates* in the sense described in [47]. That is, each of them determines the other one as follows (see [47, p. 5]):

$$\beta = GL_f\varepsilon \circ G\alpha G \circ \eta L_f G; \quad \alpha = \varepsilon L_{Ff} F \circ F\beta F \circ FL_f \eta,$$

where η and ε are the unit and the counit of the adjunction.

Theorem 5.2. *Let $F : \mathcal{C} \rightleftarrows \mathcal{C}^T : U$ be the Eilenberg–Moore factorization of a monad T on a category \mathcal{C} . Let f be a morphism in \mathcal{C} such that L_f exists.*

- (i) *There is a natural isomorphism $L_f U \cong UL_{Ff}$ if and only if T preserves f -equivalences.*
- (ii) *Suppose that T preserves f -equivalences and L_{Tf} also exists. Then the following statements are equivalent:*
 - (a) *T preserves Tf -equivalences.*
 - (b) *There is a natural isomorphism $L_f U \cong L_{Tf} U$.*
 - (c) *There is a natural isomorphism $L_{Tf} \cong L_{TTf}$.*

Proof. Suppose first that T preserves f -equivalences. Then Theorem 4.2 implies that L_f induces a localization L' on \mathcal{C}^T such that $L_f U \cong UL'$ naturally and U preserves and reflects local objects and equivalences. Hence the L' -local objects are those (X, a) in \mathcal{C}^T such that X is f -local. But Theorem 5.1 tells

us that X is f -local if and only if (X, a) is Ff -local. Hence L' is indeed an Ff -localization (which therefore exists). Conversely, the existence of L_{Ff} and the natural isomorphism $L_f U \cong UL_{Ff}$ implies that T preserves f -equivalences, according to Theorem 4.2. This proves (i).

Now assume that T preserves both f -equivalences and Tf -equivalences, and that L_{Tf} exists. Then part (i) yields natural isomorphisms

$$L_f U \cong UL_{Ff} \quad \text{and} \quad L_{Tf} U \cong UL_{FTf}.$$

Since T preserves f -equivalences, Tf is an f -equivalence. Hence all f -local objects are Tf -local, and it follows that every Ff -local object is FTf -local, since U preserves and reflects local objects in each case. Conversely, since $T = UF$, the fact that F is a retract of FUF by (2.2) implies that Ff is an FTf -equivalence, and consequently every FTf -local object is Ff -local. Thus we conclude that $L_{Ff} \cong L_{FTf}$, as claimed.

Next, the isomorphism $L_{Tf} \cong L_{TTf}$ follows from the fact that TTf is a Tf -equivalence because T preserves Tf -equivalences, and Tf is a TTf -equivalence because T is a retract of TT . Finally, each of the natural isomorphisms $L_{Tf} U \cong UL_{FTf}$ or $L_{Tf} \cong L_{TTf}$ implies that T preserves Tf -equivalences (using Theorem 4.2 in the first case). \square

If T is an idempotent monad on a category \mathcal{C} , and we denote by $I: \mathcal{S} \rightarrow \mathcal{C}$ the inclusion of the full subcategory of T -local objects and by $K: \mathcal{C} \rightarrow \mathcal{S}$ its left adjoint, then, as a special case of Theorem 5.2, we infer, for a morphism f of \mathcal{C} , the following facts:

- (i) If L_f preserves \mathcal{S} , then there is a natural isomorphism $L_f I \cong IL_{Kf}$.
- (ii) If L_{Tf} also preserves \mathcal{S} , there is a natural isomorphism $L_f I \cong L_{Tf} I$.

In fact, the proof is easier, since the counit ε of the adjunction is now an isomorphism and hence $K \cong KIK$.

Example 5.3. As an example, let T be abelianization in the category of groups. Since all localizations preserve abelian groups, part (ii) tells us that

$$L_f A \cong L_{f_{\text{ab}}} A$$

for every group homomorphism f and all abelian groups A . This fact was used in [22].

Example 5.4. As observed after Example 4.4, if $\mathcal{C}(T-, -)$ depends functorially on $\mathcal{C}(-, -)$, then T preserves f -equivalences for *every* morphism f . Therefore, the assumptions that T preserves f -equivalences and Tf -equivalences in Theorem 5.2 are automatically fulfilled. This is the case, for instance, if \mathcal{C} is the category of abelian groups and $TA = R \otimes A$, where R is a ring with 1. Thus we infer from Theorem 5.2 that there is a natural isomorphism

$$L_f M \cong L_{R \otimes f} M \tag{5.2}$$

for every left R -module M and all morphisms f of abelian groups.

Theorem 5.2 also tells us that there is no ambiguity in the right-hand term of (5.2), as it may indistinctly mean the underlying abelian group of the localization of M with respect to $R \otimes f$ in the category of R -modules or the localization of the underlying abelian group of M with respect to the morphism of abelian groups that underlies $R \otimes f$, that is,

$$L_f UM \cong U(L_{R \otimes f} M) \cong L_{U(R \otimes f)} UM.$$

6 Homotopical localizations

In the remaining sections we discuss localizations in a homotopical context, using the formalism of Quillen model categories [43].

Every model category \mathcal{M} can be equipped with *homotopy function complexes* with the properties described in [34] or [35]. In this article we choose to work with the construction of Dwyer–Kan [26, 27], namely $\mathrm{map}_{\mathcal{M}}(X, Y)$ denotes the simplicial set of morphisms from X to Y in the *hammock localization* of \mathcal{M} , which is a simplicial category. Thus, $\mathrm{map}_{\mathcal{M}}(X, Y)$ is the colimit of the nerves $NL_n^H \mathcal{M}(X, Y)$, where the objects of the category $L_n^H \mathcal{M}(X, Y)$ are strings of n morphisms in \mathcal{M} in arbitrary directions

$$X = X_0 \longleftrightarrow X_1 \longleftrightarrow X_2 \longleftrightarrow \cdots \longleftrightarrow X_{n-1} \longleftrightarrow X_n = Y, \quad (6.1)$$

where the arrows pointing backwards are weak equivalences. A morphism in $L_n^H \mathcal{M}(X, Y)$ is a commuting diagram between strings of the same type; see [25, §34 and §35] for further details.

An adjunction between model categories $F : \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 : G$ is a *Quillen adjunction* if F preserves cofibrations and trivial cofibrations or, equivalently, if G preserves fibrations and trivial fibrations. If this is the case, then they give rise to a *derived adjunction*

$$FQ : \mathrm{Ho}(\mathcal{M}_1) \rightleftarrows \mathrm{Ho}(\mathcal{M}_2) : GR \quad (6.2)$$

between the corresponding homotopy categories, after having chosen a cofibrant replacement functor Q on \mathcal{M}_1 and a fibrant replacement functor R on \mathcal{M}_2 . It then follows from [34, Proposition 16.2.1] that the induced map

$$\mathrm{map}_{\mathcal{M}_2}(FQX, Y) \longrightarrow \mathrm{map}_{\mathcal{M}_1}(X, GRY) \quad (6.3)$$

is a natural weak equivalence for all X in \mathcal{M}_1 and Y in \mathcal{M}_2 , whose value at π_0 coincides with the bijection given by the derived adjunction (6.2).

Unless otherwise specified, in the next sections we will consider this kind of simplicially enriched orthogonality between objects and maps in model categories. Thus, an object X and a map $f : A \rightarrow B$ in a model category \mathcal{M} will be called *simplicially orthogonal* (or, as in [34], *homotopically orthogonal*) if

$$\mathrm{map}_{\mathcal{M}}(f, X) : \mathrm{map}_{\mathcal{M}}(B, X) \longrightarrow \mathrm{map}_{\mathcal{M}}(A, X) \quad (6.4)$$

is a weak equivalence of simplicial sets.

The fibrant objects that are simplicially orthogonal to a given map f are called f -local, and the maps that are simplicially orthogonal to all f -local objects are called f -equivalences. An f -localization of an object X is a cofibration $l_X: X \rightarrow L_f X$ that is an f -equivalence into an f -local object. If there exists an f -localization for all X , then (L_f, l) is an idempotent monad on the homotopy category $\mathrm{Ho}(\mathcal{M})$. As such, the classes of f -equivalences and f -local objects determine each other by ordinary orthogonality in $\mathrm{Ho}(\mathcal{M})$, and a map h is an f -equivalence if and only if $L_f h$ is a weak equivalence.

If \mathcal{M} satisfies suitable assumptions—for instance, if it is left proper, cofibrantly generated and locally presentable, or left proper and cellular [34]—then L_f exists for every map f . This includes pointed or unpointed simplicial sets, Bousfield–Friedlander spectra [13] or symmetric spectra [37], groupoids [18], and many other examples. In fact, under such assumptions there exists a new model category structure on the same category \mathcal{M} , called *left Bousfield localization* with respect to f , with the same cofibrations and with the f -equivalences as weak equivalences. An f -localization $l_X: X \rightarrow L_f X$ is a fibrant replacement functor in this new model category structure; see [34, §3.3] for further details.

In what follows, a *homotopical localization* on a model category \mathcal{M} will mean an f -localization L_f for some map f . Although our use of this term here is apparently more restrictive than in previous articles (e.g. [17, 20]), the distinction only relies on set-theoretic foundations, at least if the model category \mathcal{M} is left proper and cofibrantly generated (see [17, 44]).

The next result is a homotopical version of Theorem 5.1.

Theorem 6.1. *Let $F: \mathcal{M}_1 \rightleftarrows \mathcal{M}_2: G$ be a Quillen adjunction between model categories. For a map $f: A \rightarrow B$ between cofibrant objects in \mathcal{M}_1 , the following assertions hold:*

- (i) *A fibrant object Y in \mathcal{M}_2 is Ff -local if and only if GY is f -local.*
- (ii) *F sends f -equivalences between cofibrant objects to Ff -equivalences.*
- (iii) *If L_f and L_{Ff} exist and we denote their respective units by l_1 and l_2 , then for every cofibrant X in \mathcal{M}_1 there is a homotopy unique and homotopy natural Ff -equivalence*

$$\alpha_X: FL_f X \longrightarrow L_{Ff} FX$$

such that $\alpha_X \circ F(l_1)_X \simeq (l_2)_{FX}$, and for every Y in \mathcal{M}_2 there is a homotopy unique and homotopy natural map

$$\beta_Y: L_f GY \longrightarrow GL_{Ff} Y$$

such that $\beta_Y \circ (l_1)_{GY} \simeq G(l_2)_Y$. Furthermore, α_X is a weak equivalence for all X if and only if F preserves local objects, and β_Y is a weak equivalence for all Y if and only if G preserves equivalences.

Proof. Let Y be a fibrant object of \mathcal{M}_2 and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{M}_1}(B, GY) & \xrightarrow{\mathrm{map}(f, GY)} & \mathrm{map}_{\mathcal{M}_1}(A, GY) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{map}_{\mathcal{M}_2}(FB, Y) & \xrightarrow{\mathrm{map}(Ff, Y)} & \mathrm{map}_{\mathcal{M}_2}(FA, Y), \end{array}$$

where the vertical weak equivalences are given by the adjunction. It follows that, if Y is fibrant, then GY is f -local if and only if Y is Ff -local, as claimed in part (i).

Therefore, G sends Ff -local objects to f -local objects, and this implies that F sends f -equivalences between cofibrant objects to Ff -equivalences, since

$$\mathrm{map}_{\mathcal{M}_2}(Fg, Y) \simeq \mathrm{map}_{\mathcal{M}_1}(g, GY)$$

for every f -equivalence g between cofibrant objects and every Ff -local Y .

Now (ii) implies that, for every cofibrant X in \mathcal{M}_1 , the map $F(l_1)_X$ is an Ff -equivalence. Since $L_{Ff}FX$ is Ff -local and FL_fX is cofibrant, there is a map

$$\alpha_X: FL_fX \longrightarrow L_{Ff}FX$$

such that $\alpha_X \circ F(l_1)_X \simeq (l_2)_{FX}$, and α_X is unique up to homotopy. Moreover, α_X is an Ff -equivalence since $F(l_1)_X$ and $(l_2)_{FX}$ are Ff -equivalences, and α_X is a weak equivalence if and only if FL_fX is Ff -local.

The corresponding claims for β are proved similarly. \square

Example 6.2. The model category **Gpd** of groupoids has a simplicial enrichment $N\mathrm{Hom}(-, -)$, where $\mathrm{Hom}(G, H)$ denotes the groupoid of functors $G \rightarrow H$ and N is the nerve. Since all groupoids are fibrant and cofibrant, $\mathrm{map}_{\mathbf{Gpd}}(G, H) \simeq N\mathrm{Hom}(G, H)$ for all G, H .

Fundamental groupoid and nerve form a Quillen adjunction

$$\pi : \mathbf{sSet} \rightleftarrows \mathbf{Gpd} : N$$

and Theorem 6.1 yields a morphism of groupoids

$$\alpha_X: \pi L_f X \longrightarrow L_{\pi f}(\pi X) \tag{6.5}$$

for all X and all f , which is the natural πf -equivalence studied in [18]. From the fact that (6.5) is a πf -equivalence it follows that, if X is 1-connected, then $L_{\pi f}(\pi L_f X)$ is trivial. It is a long-standing open problem to decide if $L_f X$ is in fact 1-connected for every map f when X is 1-connected.

It should be possible to extend (6.5) to higher dimensions by using a suitable category of algebraic models for n -types (for example, n -hyperc crossed complexes [15]), in which homotopical localizations can be effectively computed, as done in [18] for the model category of groupoids. This would yield relevant information on the n -type of $L_f X$, which is usually difficult to relate with the n -type of X .

Let us however emphasize that $L_{\pi f}(\pi X)$ is very different from the space $L_{P_1 f}(P_1 X)$, where P_1 denotes the first Postnikov section. The spaces $P_1 L_f X$

and $L_{P_1 f}(P_1 X)$ are not $P_1 f$ -equivalent in general. For example, let f be a map between wedges of circles such that $L_f X$ is the localization of X at the prime 3. Thus $P_1 f = f$. Let X be the Eilenberg–Mac Lane space $K(\Sigma_3, 1)$ where Σ_3 is the permutation group on three letters. Then, as shown in [21, Example 8.2], the space $L_f X = K(\Sigma_3, 1)_{(3)} \simeq K(\Sigma_3, 1)_{\widehat{3}}$ is 1-connected since Σ_3 is generated by elements of order 2, yet it is not contractible since Σ_3 has nonzero mod 3 homology. Therefore, $P_1 L_f X$ is contractible while $L_{P_1 f}(P_1 X) = L_f X$ is not. Note that, although P_1 is left adjoint to the inclusion of the full subcategory of 1-coconnected simplicial sets, the functor $L_{P_1 f}$ does not restrict to this subcategory; that is, $L_{P_1 f}(P_1 X)$ need not be a $K(G, 1)$.

7 Preservation of homotopical structures

In what follows we work with homotopy categories of algebras over monads. For this, let (T, η, μ) be a monad acting on a model category and assume that the category \mathcal{M}^T of T -algebras has a *transferred* model structure, meaning that, in the Eilenberg–Moore factorization

$$F : \mathcal{M} \rightleftarrows \mathcal{M}^T : U \quad (7.1)$$

of T , the forgetful functor U preserves and reflects weak equivalences and fibrations (compare with [5], [38, § 3], [45, Lemma 2.3], [48, § 3]).

If \mathcal{M}^T has a transferred model structure, then (7.1) is a Quillen adjunction and hence there is a derived adjunction

$$FQ : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}^T) : UR, \quad (7.2)$$

which is *not* equivalent, in general, to the Eilenberg–Moore adjunction through $\mathrm{Ho}(\mathcal{M})^T$, which can be considered if T preserves weak equivalences and hence descends to a monad on $\mathrm{Ho}(\mathcal{M})$. Counterexamples of different kinds, showing that the categories $\mathrm{Ho}(\mathcal{M}^T)$ and $\mathrm{Ho}(\mathcal{M})^T$ need not be equivalent, were discussed in [30] and [38].

We next formalize the notion that a localization (L, l) “preserves algebras” over a monad T . Our definition is motivated by results from [20], which illustrate the fact that, even in favorable cases, it is impractical to impose that LX be a T -algebra when X is a T -algebra. Instead, we impose that LX be naturally equivalent to a T -algebra, and we also need to impose that the map $l_X : X \rightarrow LX$ be naturally equivalent to a map of T -algebras if X is a T -algebra. A similar notion was considered in [48, § 3].

Definition 7.1. Let T be a monad on a model category \mathcal{M} such that \mathcal{M}^T has a transferred model structure. We say that a homotopical localization (L, l) on \mathcal{M} *preserves T -algebras* if there is a natural transformation $g : K \rightarrow L'$ of endofunctors on $\mathrm{Ho}(\mathcal{M}^T)$ together with natural isomorphisms

- (i) $h : \mathrm{Id} \rightarrow K$ in $\mathrm{Ho}(\mathcal{M}^T)$ and
- (ii) $\beta : LU \rightarrow UL'$ in $\mathrm{Ho}(\mathcal{M})$

such that $\beta \circ lU \simeq U(g \circ h)$, where $U: \text{Ho}(\mathcal{M}^T) \rightarrow \text{Ho}(\mathcal{M})$ is the forgetful functor.

It follows from this definition that, if (L, l) preserves T -algebras, then for each T -algebra X we have $LUX \simeq UL'X$ where $L'X$ is a T -algebra. Moreover, L' is equipped with a natural transformation $l': \text{Id} \rightarrow L'$, namely $l' = g \circ h$.

Lemma 7.2. *If a homotopical localization (L, l) preserves T -algebras for a monad T on a model category \mathcal{M} , then (L', l') is a localization on $\text{Ho}(\mathcal{M}^T)$.*

Proof. In order to prove that (L', l') is an idempotent monad, it suffices to check that $l'L'$ and $L'l'$ are isomorphisms on $\text{Ho}(\mathcal{M}^T)$. On one hand, $\beta L' \circ lUL' \simeq Ul'L'$. Moreover, $lLU \simeq lUL'$ and lL is an isomorphism in $\text{Ho}(\mathcal{M})$, so $Ul'L'$ is also an isomorphism. Since U reflects isomorphisms, $l'L'$ is an isomorphism. On the other hand, by naturality of β ,

$$\beta L' \circ LUl' \simeq UL'l' \circ \beta,$$

where $LUl' \simeq L\beta \circ lLU$. Since β and Ll are isomorphisms, it follows that LUl' and hence also $UL'l'$ are isomorphisms. Again, since U reflects isomorphisms, $L'l'$ is an isomorphism. \square

Example 7.3. Let P be a (possibly coloured) operad valued in simplicial sets acting on a simplicial monoidal model category \mathcal{M} , and let $P_\infty \rightarrow P$ be a cofibrant approximation of P for the Berger–Moerdijk model structure [7]. Let X be a P_∞ -algebra such that $X(c)$ is cofibrant for each colour c . Suppose that a localization L_f exists on \mathcal{M} for a map f , and suppose that $(l_X)_{c_1} \otimes \cdots \otimes (l_X)_{c_n}$ is an f -equivalence for every n -tuple of colours c_1, \dots, c_n . Then $L_f X$ admits a homotopy unique P_∞ -algebra structure such that $l_X: X \rightarrow L_f X$ is a map of P_∞ -algebras, according to [20, Theorem 6.1].

Furthermore, specializing to the case when \mathcal{M} is the category of symmetric spectra with the positive stable model structure, [20, Theorem 7.3] states the following. Let X be a P -algebra such that $X(c)$ is cofibrant for each colour c . For a map f , if $(l_X)_{c_1} \wedge \cdots \wedge (l_X)_{c_n}$ is an f -equivalence for every n -tuple of colours c_1, \dots, c_n , then there is a P -algebra KX naturally weakly equivalent to X together with a map $KX \rightarrow L'X$ of P -algebras naturally weakly equivalent to $l_X: X \rightarrow L_f X$. This general result was used to infer in [20, Theorem 7.7] that f -localizations preserve modules over ring spectra, not just up to homotopy, but in a strict sense.

Theorem 7.4. *Let T be a monad on a model category \mathcal{M} where localizations with respect to maps exist, and suppose that \mathcal{M}^T has a transferred model structure. Let f be a map between cofibrant objects in \mathcal{M} such that L_f preserves T -algebras. If $T = UF$ is the Eilenberg–Moore factorization of T , then the following hold:*

- (i) *The natural map $\beta: L_f UX \rightarrow UL_{Ff} X$ is a weak equivalence for every T -algebra X .*

(ii) If L_{Tf} preserves T -algebras and T preserves cofibrant objects, then

$$\beta: L_f UX \longrightarrow L_{Tf} UX$$

is a weak equivalence for every T -algebra X .

Proof. Since L_f preserves T -algebras, there is a functor

$$L': \mathrm{Ho}(\mathcal{M}^T) \longrightarrow \mathrm{Ho}(\mathcal{M}^T)$$

together with a natural transformation $l': \mathrm{Id} \rightarrow L'$ and a natural isomorphism $\beta: L_f U \rightarrow UL'$ such that $\beta \circ lU \simeq Ul'$. We are going to show that L' is an Ff -localization. In view of Lemma 7.2, we only need to prove that the classes of L' -local objects and Ff -local objects coincide. By Proposition 6.1, an object X is Ff -local if and only if UX is f -local. This is the case if and only if l_{UX} is an isomorphism, and from the hypotheses it follows that l_{UX} is an isomorphism if and only if Ul'_X is an isomorphism. Since U reflects isomorphisms, this happens if and only if X is L' -local, as we wanted to prove.

Therefore, the natural map $\beta: L_f UX \rightarrow UL_{Ff} X$ is a weak equivalence for every T -algebra X , hence yielding (i).

Consequently, T preserves f -equivalences between cofibrant objects, since $L_f Tg = L_f U Fg \simeq UL_{Ff} Fg$ and F sends f -equivalences between cofibrant objects to Ff -equivalences by part (ii) of Proposition 6.1.

Next, from part (i) we obtain that

$$L_f UX \simeq UL_{Ff} X \quad \text{and} \quad L_{Tf} UX \simeq UL_{FTf} X$$

for every T -algebra X . The rest of the argument is the same as in the proof of Theorem 5.2. Since T preserves f -equivalences between cofibrant objects, Tf is an f -equivalence and hence every f -local object is Tf -local. Therefore, every Ff -local object is FTf -local, by part (i) of Proposition 6.1. Conversely, since Ff is a retract of FTf , it follows that Ff is an FTf -equivalence, and hence all FTf -local objects are Ff -local. \square

Example 7.5. Let \mathbf{Sp} be the category of symmetric spectra with the positive stable model structure [46]. For a ring spectrum E , the category $E\text{-}\mathbf{Mod}$ of left E -modules also admits a model structure [46, Theorem 2.6] and there is a Quillen adjunction

$$F: \mathbf{Sp} \rightleftarrows E\text{-}\mathbf{Mod} : U$$

where $FX = E \wedge X$ and U is the forgetful functor, which preserves and reflects weak equivalences. This is in fact the Eilenberg–Moore factorization of the monad $TX = U(E \wedge X)$ on \mathbf{Sp} , and T preserves cofibrant objects if E is cofibrant.

It follows from Theorem 7.4 that, if E is any cofibrant ring spectrum and f is any map between cofibrant spectra, then, assuming that E is connective or that L_f commutes with suspension —according to [20], either one of these conditions implies that L_f preserves T -algebras; moreover, if L_f commutes with suspension, then L_{Tf} also commutes with suspension— then:

(i) $L_f UM \simeq UL_{E \wedge f} M$, and

(ii) $L_f UM \simeq L_{U(E \wedge f)} UM$

for all left E -modules M .

As in (5.2), we conclude that, if E is connective or L_f commutes with suspension, then

$$L_f M \simeq L_{E \wedge f} M \quad (7.3)$$

for all left E -modules M , in all possible readings of this expression.

It follows from this result that, if X is any cofibrant spectrum and L_X denotes Bousfield localization with respect to X_* -homology, then for every cofibrant ring spectrum E the functors L_X and $L_{E \wedge X}$ coincide on left E -modules. The case $E = H\mathbb{Z}$ is especially relevant, since it allows a complete description of all homological localizations of stable GEMs; see [30].

In fact, the coincidence of L_X and $L_{E \wedge X}$ on left E -modules was shown for homotopy ring spectra and homotopy modules in [31, Proposition 3.2], while we have proved it here for strict ring spectra and strict modules. This prompts a need to inspect the validity of Theorem 7.4 for *homotopy* T -algebras, and we do so in Section 9.

8 Operads and monads

In what follows, Ω denotes the derived loop functor on pointed simplicial sets, i.e., $\Omega X = \text{Map}_*(\mathbb{S}^1, RX)$, where R is a fibrant replacement functor.

Recall that for every operad P valued in simplicial sets acting on a simplicial monoidal model category \mathcal{M} —here we are implicitly assuming, as usual, that \mathcal{M} is closed symmetric monoidal—there is a monad T on \mathcal{M} such that the T -algebras and the P -algebras coincide, namely

$$TX = \coprod_{n \geq 0} P(n) \otimes_{\Sigma_n} X^{\otimes n} \quad (8.1)$$

if P is symmetric, or correspondingly without the Σ_n action otherwise.

If the operad P is unital (i.e., if $P(0)$ is the unit of the monoidal structure on \mathcal{M}), then the algebras over (8.1) coincide with the algebras over the corresponding *reduced* monad, which is obtained from (8.1) by identifying $(p, s_i y)$ with $(\sigma_i p, y)$ for $1 \leq i \leq n$ and every $p \in P(n)$ and $y \in X^{n-1}$, where $s_i: X^{n-1} \rightarrow X^n$ inserts the unit in the i -th place and $\sigma_i: P(n) \rightarrow P(n-1)$ is the i -th degeneracy; see [41, §4] for details.

Let A be the (non-symmetric) associative operad and let $\varphi: A_\infty \rightarrow A$ be a cofibrant approximation. Then the algebras over A in pointed simplicial sets are the monoids and the associated reduced monad is the James (free monoid) construction.

The map of operads $\varphi: A_\infty \rightarrow A$ yields a Quillen equivalence

$$\varphi_! : A_\infty\text{-sSet}_* \rightleftarrows \mathbf{Mon} : \varphi^*,$$

so the homotopy category of monoids is equivalent to the homotopy category of A_∞ -algebras in pointed simplicial sets. Moreover, the classifying space functor B is part of an adjunction

$$B : \mathrm{Ho}(A_\infty\text{-}\mathbf{sSet}_*) \rightleftarrows \mathrm{Ho}(\mathbf{sSet}_*) : \Omega,$$

which restricts, as in (2.3), to an equivalence of categories between the full subcategory of $\mathrm{Ho}(A_\infty\text{-}\mathbf{sSet}_*)$ whose objects are those M such that the unit $\eta_M : M \rightarrow \Omega BM$ is an isomorphism (that is, the grouplike monoids) and the full subcategory of connected simplicial sets, which are precisely those X for which the counit $\varepsilon_X : B\Omega X \rightarrow X$ is an isomorphism. Both subcategories are preserved by f -localizations, since a monoid M is grouplike if and only if $\pi_0(M)$ is a group, and $\pi_0(L_f X) \cong \pi_0(X)$ for every X if L_f is nontrivial.

Theorem 8.1. *If f is any map of connected simplicial sets, then*

- (i) $L_f \Omega X \simeq \Omega L_{\Sigma f} X$, and
- (ii) $L_f \Omega X \simeq L_{\Omega \Sigma f} \Omega X$

for all pointed simplicial sets X .

Proof. Consider the derived adjunction

$$J : \mathrm{Ho}(\mathbf{sSet}_*) \rightleftarrows \mathrm{Ho}(A_\infty\text{-}\mathbf{sSet}_*) : U,$$

where J is the James functor. By [20, Theorem 6.1], L_f preserves A_∞ -algebras. Therefore, since U reflects isomorphisms, it follows from Theorem 7.4 that $L_f U M \simeq U L_{Jf} M$ for all M .

Now, given a simplicial set X , if we view ΩX as an A_∞ -algebra, we obtain

$$L_f U \Omega X \simeq U L_{Jf} \Omega X \simeq U L_{Jf} \Omega B \Omega X,$$

and, since equivalences of categories commute with localizations,

$$U L_{Jf} \Omega(B \Omega X) \simeq U \Omega L_{BJf} B \Omega X \simeq U \Omega L_{\Sigma f} B \Omega X \simeq U \Omega L_{\Sigma f} X,$$

where we used the fact that $JY \simeq \Omega \Sigma Y$ if Y is connected, and also the fact that $L_{\Sigma f}$ preserves connected components.

It then follows from part (ii) of Theorem 7.4 that

$$L_f U \Omega X \simeq L_{UJf} U \Omega X \simeq L_{\Omega \Sigma f} U \Omega X,$$

as claimed in (ii). \square

By induction we also have $L_f \Omega^n X \simeq \Omega^n L_{\Sigma^n f} X$ for all X and $n \geq 0$. This formula also holds for $n = \infty$, by the following argument, which is analogous to the preceding one.

Let E be the commutative operad and $\psi : E_\infty \rightarrow E$ a cofibrant approximation. The algebras over E in pointed simplicial sets are the commutative monoids and the associated reduced monad is the infinite symmetric product SP^∞ ; cf. [23]. Connected commutative monoids are GEMs, that is, homotopy

equivalent to products of Eilenberg–Mac Lane spaces $K(G, n)$ with G abelian when $n = 1$.

For a space X , the quotient X^n/Σ_n by the symmetric group action does not have the same homotopy type for $n \geq 2$ as $C(n) \times_{\Sigma_n} X^n$, where $C(n)$ is a contractible space with a free Σ_n -action. For this reason, E_∞ -spaces are not homotopy equivalent to commutative monoids, in general. Instead, if we denote by $B^\infty X$ the Ω -spectrum associated with a given E_∞ -space X , then there is an adjunction

$$B^\infty : \mathrm{Ho}(E_\infty\text{-}\mathbf{sSet}_*) \rightleftarrows \mathrm{Ho}(\mathbf{Sp}) : \Omega^\infty,$$

which restricts, as another instance of (2.3), to an equivalence of categories between the full subcategory of grouplike E_∞ -spaces (i.e., infinite loop spaces) and the full subcategory of connective spectra; see [3, Pretheorem 2.3.2].

If C is the reduced monad associated with an E_∞ -operad, then May's Approximation Theorem [40] implies that $CX \simeq \Omega^\infty \Sigma^\infty X$ if X is connected.

Theorem 8.2. *If f is any map of connected simplicial sets, then*

- (i) $L_f \Omega^\infty X \simeq \Omega^\infty L_{\Sigma^\infty f} X$, and
- (ii) $L_f \Omega^\infty X \simeq L_{\Omega^\infty \Sigma^\infty f} \Omega^\infty X$

for every spectrum X .

Proof. Using Theorem 7.4 and [20, Theorem 6.1] as in the proof of the previous theorem, we obtain the following equivalences, where X^c denotes the connective cover of X and $QX = \Omega^\infty \Sigma^\infty X$:

$$\begin{aligned} L_f U \Omega^\infty X &\simeq U L_{Qf} \Omega^\infty X \simeq U L_{Qf} \Omega^\infty X^c \\ &\simeq U \Omega^\infty L_{B^\infty Qf} X^c \simeq U \Omega^\infty L_{\Sigma^\infty f} X^c \simeq U \Omega^\infty L_{\Sigma^\infty f} X, \end{aligned}$$

since $\Sigma^\infty f$ is a map of connective spectra and thus $L_{\Sigma^\infty f} X^c \simeq (L_{\Sigma^\infty f} X)^c$. \square

9 Homotopy algebras

Given a monad (T, η, μ) on a model category \mathcal{M} such that T preserves weak equivalences, we call *homotopy T -algebras* the objects of the Eilenberg–Moore category $\mathrm{Ho}(\mathcal{M})^T$. Thus a homotopy T -algebra is an object X in \mathcal{M} equipped with a map $a : TX \rightarrow X$ such that $a \circ \eta_X \simeq \mathrm{id}$ and $a \circ \mu_X \simeq a \circ Ta$. Our aim in this section is to prove that the equivalence $L_f X \simeq L_{Tf} X$ obtained in part (ii) of Theorem 7.4 also holds when X is a homotopy T -algebra, provided that T and f interact in a convenient way.

Theorem 9.1. *Let \mathcal{M} be a model category and let (T, η, μ) be a monad on \mathcal{M} preserving weak equivalences. If f is a map in \mathcal{M} such that L_f and L_{Tf} exist, and T preserves f -equivalences and Tf -equivalences, then*

$$L_f X \simeq L_{Tf} X$$

if X underlies a homotopy T -algebra.

Proof. Consider the Eilenberg–Moore factorization

$$F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M})^T : U.$$

Since T preserves f -equivalences, it follows from Theorem 4.2 that there is a localization L' on $\mathrm{Ho}(\mathcal{M})^T$ such that U preserves and reflects local objects and equivalences, and moreover $L_f UX \simeq UL'X$ naturally for every homotopy T -algebra X . Since T also preserves Tf -equivalences, we infer similarly that $L_{Tf} UX \simeq UL''X$ for a localization L'' . We next show that, as in the proof of Theorem 5.2, we have $L' \simeq L''$.

The L' -local objects are those X in $\mathrm{Ho}(\mathcal{M})^T$ such that UX is f -local, and the L'' -local objects are those such that UX is Tf -local. Since T preserves f -equivalences, Tf is an f -equivalence and hence every f -local object of \mathcal{M} is Tf -local. This tells us that all L' -local objects of $\mathrm{Ho}(\mathcal{M})^T$ are L'' -local. To prove the converse, suppose that X is Tf -local. Then $\mathrm{map}_{\mathcal{M}}(Tf, X)$ is a weak equivalence; hence, we will have achieved our goal if we prove that $\mathrm{map}_{\mathcal{M}}(f, X)$ is a homotopy retract of $\mathrm{map}_{\mathcal{M}}(Tf, X)$. For this, let η be the unit of T and consider the composite

$$\mathrm{map}_{\mathcal{M}}(f, X) \xrightarrow{t} \mathrm{map}_{\mathcal{M}}(Tf, TX) \xrightarrow{a_*} \mathrm{map}_{\mathcal{M}}(Tf, X) \xrightarrow{(\eta_f)^*} \mathrm{map}_{\mathcal{M}}(f, X),$$

where $a : TX \rightarrow X$ is the structure map of X as a homotopy T -algebra (so $a \circ \eta_X \simeq \mathrm{id}$), and t is defined as follows. The map t is induced by the functor τ which sends each string of morphisms as in (6.1) to the string obtained by applying T termwise. Since η is a natural transformation, it yields a homotopy $(\eta_X)_* \simeq (\eta_f)^* \circ t$, where $(\eta_X)_*$ is the map of nerves induced by the right-composition functor with $\eta_X : X \rightarrow TX$ and $(\eta_f)^*$ is induced by left-composition with $\eta_f : f \rightarrow Tf$. Since a_* and $(\eta_f)^*$ commute, the composite $(\eta_f)^* \circ a_* \circ t$ is homotopic to $a_* \circ (\eta_X)_*$, which is in its turn homotopic to the identity since $a \circ \eta_X$ is homotopic to the identity —see [44] for additional details about the fact that homotopic morphisms in \mathcal{M} induce homotopic maps of homotopy function complexes.

This argument proves that $\mathrm{map}_{\mathcal{M}}(f, X)$ is indeed a homotopy retract of $\mathrm{map}_{\mathcal{M}}(Tf, X)$ and hence $\mathrm{map}_{\mathcal{M}}(f, X)$ is also a weak equivalence, so X is f -local, as needed. \square

As we next show, the condition that T preserves both f -equivalences and Tf -equivalences in Theorem 9.1 is automatically satisfied for every f if T comes from an operad as in (8.1).

Lemma 9.2. *Let T be the reduced or unreduced monad associated with a unital operad P acting on pointed simplicial sets. Then T preserves f -equivalences for every map f .*

Proof. As shown in [24, 1.G], the functor L_f preserves finite products for every f . Hence, if $g : X \rightarrow Y$ is an f -equivalence, then so is the induced map

$$W \times X \times \cdots \times X \longrightarrow W \times Y \times \cdots \times Y$$

for every finite number of factors and every simplicial set W . This is enough if the operad P is non-symmetric. If P is symmetric, observe that the quotient $P(n) \times_{\Sigma_n} X^n$ is a colimit over Σ_n (viewed as a small category), and it is also a homotopy colimit if Σ_n acts freely on $P(n)$, but not otherwise. However, as explained in [24, 4.A], $P(n) \times_{\Sigma_n} X^n$ is a homotopy colimit of a (free) diagram indexed by the opposite of the orbit category of Σ_n , where the value of the diagram at Σ_n/G is the fixed-point space $P(n)^G \times (X^n)^G$. Since each space $(X^n)^G$ is homeomorphic to a product X^m with $m \leq n$, we obtain that the map $Tg: TX \rightarrow TY$ is a homotopy colimit of f -equivalences, and therefore it is itself an f -equivalence because, as in (3.1), the comparison map

$$\alpha: \operatorname{hocolim}_{i \in I} L_f X_i \longrightarrow L_f \left(\operatorname{hocolim}_{i \in I} X_i \right)$$

is an f -equivalence for every small category I . \square

Corollary 9.3. *If T is the reduced or unreduced monad associated with a unital operad acting on pointed simplicial sets, then, for every map f , we have*

$$L_f X \simeq L_{Tf} X$$

if X is the underlying space of a homotopy T -algebra.

Proof. This is an immediate consequence of Theorem 9.1 and Lemma 9.2. \square

Example 9.4. The infinite symmetric product SP^∞ is the reduced monad associated with the commutative operad acting on pointed simplicial sets. As shown in [22, Proposition 1.1], the homotopy algebras over SP^∞ coincide with the strict algebras —this is parallel to the fact that, in stable homotopy, the classes of homotopy $H\mathbb{Z}$ -module spectra and strict $H\mathbb{Z}$ -module spectra coincide, as proved in [31].

Corollary 9.3 implies that $L_f X \simeq L_{SP^\infty f} X$ for every GEM X , as already pointed out in [22, Theorem 1.3]. Hence, in particular,

$$L_f SP^\infty X \simeq L_{SP^\infty f} SP^\infty X$$

for every space X .

Example 9.5. The previous example can be generalized as follows. For every cofibrant ring spectrum E , consider the monad $TX = \Omega^\infty(E \wedge \Sigma^\infty X)$ on $\operatorname{Ho}(\mathbf{sSet}_*)$. We call its homotopy algebras *unstable E -modules*. If E is connective or L_f commutes with suspension, then $E \wedge -$ preserves f -equivalences and Tf -equivalences. Then Theorem 9.1 also applies and yields

$$L_f X \simeq L_{\Omega^\infty(E \wedge \Sigma^\infty f)} X$$

for all unstable E -modules X . The special case $E = H\mathbb{Z}$ is Example 9.4, and the case $E = S$ (the sphere spectrum) is discussed in the next example.

Example 9.6. The homotopy algebras over the James functor J are the monoids in the Cartesian monoidal category $\operatorname{Ho}(\mathbf{sSet}_*)$, that is, the homotopy associative H -spaces. Therefore, by Corollary 9.3, we have $L_f X \simeq L_{\Omega \Sigma f} X$ for every

homotopy associative H -space X and every map f between connected spaces. This is a more general statement than part (ii) of Theorem 8.1.

It follows similarly that $L_f X \simeq L_{\Omega^\infty \Sigma^\infty f} X$ for every map f between connected spaces and every homotopy algebra X over the reduced monad C associated with an E_∞ -operad. The homotopy C -algebras are the H_∞ -spaces in the sense of [14, I.3.7]. As shown in [38, 42], H_∞ -spaces need not be homotopy equivalent to E_∞ -spaces. Thus we have also sharpened part (ii) of Theorem 8.2.

10 Colocalizations

In this section we state duals of the main results of the previous sections. Proofs are omitted. A full subcategory \mathcal{S} of a category \mathcal{C} is *coreflective* if the inclusion I has a right adjoint $I : \mathcal{S} \rightleftarrows \mathcal{C} : K$. Then the functor $C = IK$ is called a *coreflection* or a *colocalization* on \mathcal{C} . In this case, the unit $\text{Id} \rightarrow KI$ is an isomorphism. We denote the counit by $c : C \rightarrow \text{Id}$. An object of \mathcal{C} is called *C -colocal* if it is isomorphic to an object in the subcategory \mathcal{S} ; hence, X is C -colocal if and only if $c_X : CX \rightarrow X$ is an isomorphism. A morphism $g : U \rightarrow V$ is a *C -equivalence* if Cg is an isomorphism, or, equivalently, if for all C -colocal objects X composition with g induces a bijection

$$\mathcal{C}(X, U) \cong \mathcal{C}(X, V). \quad (10.1)$$

Conversely, the C -colocal objects are precisely those X for which (10.1) holds for all C -equivalences $g : U \rightarrow V$.

Theorem 10.1. *Let $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a functor. Let C_1 be a colocalization on \mathcal{C}_1 with counit c_1 , and C_2 a colocalization on \mathcal{C}_2 with counit c_2 . Then the following hold:*

- (i) *F preserves colocal objects if and only if there is a natural transformation*

$$\alpha : FC_1 \longrightarrow C_2F$$

such that $c_2F \circ \alpha = FC_1$. If this holds, then α is unique, and it is an isomorphism if and only if F preserves equivalences.

- (ii) *F preserves equivalences if and only if there is a natural transformation*

$$\beta : C_2F \longrightarrow FC_1$$

such that $FC_1 \circ \beta = c_2F$ and β_X is a C -equivalence for all X . If this holds, then β is unique, and it is an isomorphism if and only if F preserves colocal objects.

Theorem 10.2. *Let (T, η, μ) be a monad on a category \mathcal{C} and let (C, c) be a colocalization on \mathcal{C} . Then the following statements are equivalent:*

- (a) *T preserves C -colocal objects.*

- (b) For every T -algebra (X, a) there is a unique T -algebra structure on CX such that $c_X: CX \rightarrow X$ is a morphism of T -algebras.
- (c) There is a colocalization C' on the category \mathcal{C}^T of T -algebras such that $CU \cong UC'$ naturally, where U denotes the forgetful functor.
- (d) There is a colocalization C' on the category \mathcal{C}^T of T -algebras such that the forgetful functor U preserves and reflects colocal objects and equivalences.

A morphism $g: U \rightarrow V$ and an object A in a category \mathcal{C} are called *co-orthogonal* if the function

$$\mathcal{C}(A, g): \mathcal{C}(A, U) \longrightarrow \mathcal{C}(A, V) \quad (10.2)$$

is a bijection. The morphisms co-orthogonal to a given object A are called *A-cellular equivalences* or more shortly *A-equivalences*, and the objects co-orthogonal to all A -equivalences are called *A-cellular*. An *A-cellularization* (or *A-colocalization*) of an object X is an A -cellular equivalence from an A -cellular object $c_X: C_A X \rightarrow X$. If an A -cellularization exists for all objects, then C_A is indeed a colocalization on \mathcal{C} . The existence of C_A is ensured for all objects A if the category \mathcal{C} is locally presentable.

Proposition 10.3. *Let $F: \mathcal{C}_1 \rightleftarrows \mathcal{C}_2: G$ be a pair of adjoint functors. For an object A in \mathcal{C}_1 , the following assertions hold:*

- (i) *A morphism g in \mathcal{C}_2 is an FA -equivalence if and only if Gg is an A -equivalence.*
- (ii) *F sends A -cellular objects to FA -cellular objects.*
- (iii) *If C_A and C_{FA} exist, then there is a unique natural transformation*

$$\alpha: FC_A \longrightarrow C_{FA}F$$

such that $c_{FX} \circ \alpha_X = Fc_X$ for all X . There is also a unique natural transformation

$$\beta: C_A G \longrightarrow G C_{FA}$$

such that $Gc_Y \circ \beta_Y = c_{GY}$ and β_Y is an A -equivalence for all X . Furthermore, α is an isomorphism if and only if F preserves cellular equivalences, and β is an isomorphism if and only if G preserves cellular objects.

Theorem 10.4. *Let $F: \mathcal{C} \rightleftarrows \mathcal{C}^T: U$ be the Eilenberg–Moore factorization of a monad T on a category \mathcal{C} . Let A be an object in \mathcal{C} such that C_A exists.*

- (i) *There is a natural isomorphism $C_A U \cong U C_{FA}$ if and only if T preserves A -cellular objects.*
- (ii) *Suppose that T preserves A -equivalences and C_{TA} exists. Then the following statements are equivalent:*

- (a) T preserves TA -cellular objects.
- (b) There is a natural isomorphism $C_A U \cong C_{TA} U$.
- (c) There is a natural isomorphism $C_{TA} \cong C_{TTA}$.

Example 10.5. Let \mathcal{C} be the category of abelian groups and $TA = R \otimes A$, where R is a ring with 1. We infer from Theorem 10.4 that there is a natural isomorphism

$$C_A M \cong C_{R \otimes A} M \quad (10.3)$$

for every left R -module M and all abelian groups A .

Theorem 10.4 also tells us that there is no ambiguity in the right-hand term of (10.3), as it may indistinctly mean the underlying abelian group of the cellularization of M with respect to $R \otimes A$ in the category of R -modules or the cellularization of the underlying abelian group of M with respect to the abelian groups that underlies $R \otimes A$, that is,

$$C_A U M \cong U(C_{R \otimes A} M) \cong C_{U(R \otimes A)} U M.$$

Proposition 10.6. *Let $F : \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 : G$ be a Quillen adjunction between model categories. For a cofibrant object A in \mathcal{M}_1 , the following assertions hold:*

- (i) *A morphism g in \mathcal{M}_2 is an FA -equivalence if and only if Gg is A -equivalence.*
- (ii) *F sends A -cellular objects to FA -cellular objects.*
- (iii) *If C_A and C_{FA} exist, then for every cofibrant X in \mathcal{M}_1 there is a homotopy unique and homotopy natural map*

$$\alpha_X : FC_A X \longrightarrow C_{FA} F X$$

such that $c_{FX} \circ \alpha_X = Fc_X$, and for every fibrant Y in \mathcal{M}_2 there is also a homotopy unique A -equivalence

$$\beta_Y : C_A G Y \longrightarrow G C_{FA} Y$$

such that $Gc_Y \circ \beta_Y = c_{GY}$. Furthermore, α_X is a weak equivalence for all X if and only if F preserves cellular equivalences, and β_Y is a weak equivalence for all Y if and only if G preserves cellular objects.

Theorem 10.7. *Let T be a monad on a model category \mathcal{M} where cellularizations with respect to objects exist, and suppose that \mathcal{M}^T has a transferred model structure. Let A be a cofibrant object in \mathcal{M} such that C_A preserves T -algebras. If $T = UF$ is the Eilenberg–Moore factorization of T , then the following hold:*

- (i) *The natural map $\beta : C_A U X \rightarrow U C_{FA} X$ is a weak equivalence for every T -algebra X .*

(ii) If C_{TA} preserves T -algebras and T preserves cofibrant objects, then

$$\beta: C_A UX \longrightarrow C_{TA} UX$$

is a weak equivalence for every T -algebra X .

Example 10.8. Let E be a cofibrant ring spectrum and let A be any spectrum. Consider the Quillen adjunction

$$F: \mathbf{Sp} \rightleftarrows E\text{-}\mathbf{Mod} : U$$

where $FX = E \wedge X$ and U is the forgetful functor. If E is connective or C_A commutes with suspension, then

(i) $C_A UM \simeq UC_{E \wedge A} M$, and

(ii) $C_A UM \simeq C_{U(E \wedge A)} UM$

for all left E -modules M .

To prove this, recall from [32, Theorem 5.8] that if \mathcal{M} is a monoidal model category, C_A is a cellularization and M is a (strict) P -algebra over a cofibrant operad P such that the unit is A -cellular, and for every $n \geq 1$ the object $C_A M(c_1) \otimes \cdots \otimes C_A M(c_n)$ is A -cellular whenever $P(c_1, \dots, c_n; c) \neq 0$, then $C_A M$ admits a homotopy unique P -algebra structure such that c_M is a map of P -algebras. From this fact it follows that $c_M: C_A M \rightarrow M$ is naturally weakly equivalent to an E -module map $C'_M \rightarrow KM$ where $KM \simeq M$ as E -modules. Then it follows as in the case of localizations that $C_A UM \simeq UC_{E \wedge A} M$ for every left E -module M , hence yielding (i). Part (ii) is also proved as in the dual case.

As in (5.2), we conclude that

$$C_A M \simeq C_{E \wedge A} M \tag{10.4}$$

for all left E -modules M and all spectra A , in all possible readings of this expression.

Theorem 10.9. *If A is any connected simplicial set, then*

(i) $C_A \Omega X \simeq \Omega C_{\Sigma A} X$, and

(ii) $C_A \Omega X \simeq C_{\Omega \Sigma A} \Omega X$

for all pointed simplicial sets X , and

(iii) $C_A \Omega^\infty Y \simeq \Omega^\infty C_{\Sigma^\infty A} Y$, and

(iv) $C_A \Omega^\infty Y \simeq C_{\Omega^\infty \Sigma^\infty A} \Omega^\infty Y$

for every spectrum Y .

Part (i) appeared in Farjoun's book [24], while the other formulas are new, as far as we know.

Theorem 10.10. *Let \mathcal{M} be a model category and let T be a monad on \mathcal{M} preserving weak equivalences. If A is an object in \mathcal{M} such that C_A and C_{TA} exist, and T preserves A -equivalences and TA -equivalences, then*

$$C_A X \simeq C_{TA} X$$

if X underlies a homotopy T -algebra.

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